J. Fluid Mech. (1970), *vol.* 43, *part* 3, *pp.* 433–449
Printed in Great Britain

Bottomless harbours

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(Received 14 January 1970)

Does the harbour of an artificial island need a bottom? The excitation of waves inside a partially immersed open circular cylinder is considered. An incident plane wave is expanded in Bessel functions and for each mode the problem is formulated in terms of the radial displacement on the cylindrical interface below the cylinder. The solution is obtainable either from an infinite set of simultaneous equations or from an integral equation. It is shown that the phase of the solution is independent of depth and resonances are found at wave-numbers close to those of free oscillations in a cylinder extending to the bottom. If the resonances of the cylinder are made sharper (by increasing the depth of immersion) the peak response of the harbour increases, but the response to a continuous spectrum remains approximately constant. Numerical results are obtained by minimizing the least squares error of a finite number N of simultaneous equations. Convergence is slow, but the error is roughly proportional to $1/N$ and this is exploited. The solution obtained from a variational formulation using the incoming wave as a trial function is found to give a very good approximation for small wave-numbers, but is increasingly inaccurate for large wave-numbers. Away from resonance the amplitude of the harbour oscillation is less than 10 % of the amplitude of the incoming wave provided the depth of the cylinder is greater than about $\frac{1}{4}$ wavelength, and it is argued that in practice at the resonant wave-number oscillations excited through the bottom of the harbour will leak out through the entrance before they can reach large amplitudes. In an appendix the excitation of harbour oscillations through the harbour entrance is discussed, and some results of Miles & Munk (1961) on an alleged harbour paradox are re-interpreted.

1. Introduction

Surface gravity waves are incident on a hollow cylinder partially immersed in water of finite depth, as illustrated in figure 1. The main object of this paper is to evaluate the resulting wave motion inside the cylinder. The results will be useful in deciding whether artificial islands, such as that proposed by the Scripps Institution of Oceanography, need have bottoms to their harbours. Clearly a bottomless island is cheaper to construct than one with a bottom, but the bottom can only be omitted if the harbour will still remain sufficiently calm. The results may also help decide whether it is feasible to protect a small area of sea from swell by building a sufficiently deep 'wall' around it, enabling work inside this area to be carried out in calm water.

Mathematically the problem has much in common with that considered by Miles & Gilbert (1968). They considered the scattering of waves by a circular dock, i.e. by an artificial island with a bottom. The formulation (though not the solution) of the present problem closely follows Miles & Gilbert.

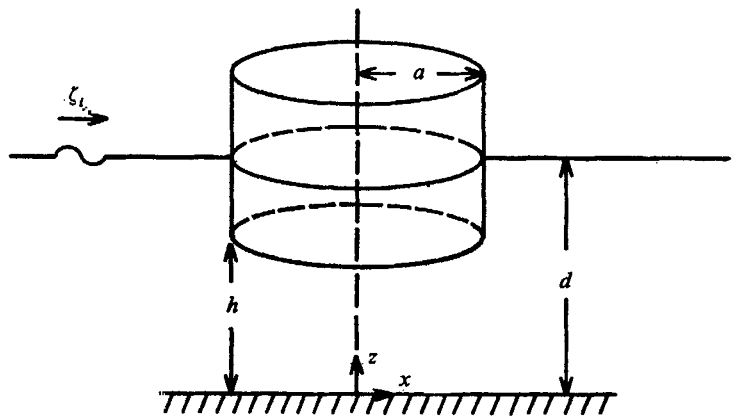

FIGURE 1. Open-bottomed circular cylinder of radius a immersed to depth $d-h$ in water of depth d.

2. Formulation

Let the incident plane wave have amplitude ζ_0, frequency σ and wavenumber k. σ and k are related through the dispersion relation

$$\sigma^2 = gk \tanh kd, \tag{2.1}$$

where d is the depth of the water. The cylinder of radius a is immersed to a depth $d-h$. We assume small amplitude irrotational flow.

The free surface displacement may be described by the real part of $\zeta e^{-i\sigma t}$, where the incoming wave has

$$\zeta = \zeta_0 e^{ikx} \tag{2.2}$$

$$= \zeta_0 \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta \tag{2.3}$$

and
$$\epsilon_0 = 1, \quad \epsilon_m = 2 \quad (m \geqslant 1). \tag{2.4}$$

In view of the nature of this cylindrical wave expansion it is appropriate to express the total disturbance as

$$\zeta(r,\theta) = \zeta_0 \sum_{m=0}^{\infty} \epsilon_m i^m \chi_m(r) \cos m\theta. \tag{2.5}$$

The corresponding displacement potential $[(1/-i\sigma) \times$ the velocity potential$]$ is

$$\phi(r,\theta,z) = \zeta_0 \sum_{m=0}^{\infty} \epsilon_m i^m \psi_m(r,z) \cos m\theta, \tag{2.6}$$

where
$$\chi_m(r) = \left.\frac{\partial \psi_m}{\partial z}\right|_{z=d} \tag{2.7}$$

in order to satisfy the kinematic free surface condition. ϕ must also satisfy

$$\nabla^2\phi = 0, \tag{2.8}$$

$$\sigma^2\phi - g\,\partial\phi/\partial z = 0 \quad \text{on} \quad z = d, \tag{2.9}$$

$$\partial\phi/\partial z = 0 \quad \text{on} \quad z = 0, \tag{2.10}$$

$$\partial\phi/\partial r = 0 \quad \text{on} \quad r = a \quad \text{for} \quad h \leqslant z \leqslant d, \tag{2.11}$$

where (2.9) arises from the dynamic free surface condition, and (2.10), (2.11) are the conditions for zero normal displacement on the rigid boundaries.

ψ_m will be composed of the general solutions of (2.8), (2.9), (2.10), namely

$$\left.\begin{aligned} &J_m(kr)\cosh kz,\\ &Y_m(kr)\cosh kz,\\ &I_m(\alpha r)\cos\alpha z,\\ &K_m(\alpha r)\cos\alpha z,\end{aligned}\right\} \tag{2.12}$$

where α is a real positive solution of

$$\alpha\tan\alpha d + \sigma^2/g = 0, \tag{2.13}$$

J_m, Y_m are ordinary Bessel functions and I_m, K_m are modified Bessel functions. Following Miles & Gilbert we introduce the functions

$$\left.\begin{aligned} Z_k(z) &= N_k^{-\frac{1}{2}}\cosh kz,\\ Z_\alpha(z) &= N_\alpha^{-\frac{1}{2}}\cos\alpha z,\end{aligned}\right\} \tag{2.14}$$

where

$$\left.\begin{aligned} N_k &= \frac{1}{2}\left[1+\frac{\sinh 2kd}{2kd}\right],\\ N_\alpha &= \frac{1}{2}\left[1+\frac{\sin 2\alpha d}{2\alpha d}\right].\end{aligned}\right\} \tag{2.15}$$

$Z_k(z)$, $Z_\alpha(z)$ form a complete orthogonal set in $[0, d]$ with mean square values of 1.

The radial displacement, and hence $\partial\psi_m/\partial r$, is continuous at $r = a$ for $0 \leqslant z < h$. Suppose that

$$\partial\psi_m/\partial r = f_m(z) \quad \text{at} \quad r = a \quad \text{for} \quad 0 \leqslant z < h, \tag{2.16}$$

and remember that from (2.11)

$$\partial\psi_m/\partial r = 0 \quad \text{at} \quad r = a \quad \text{for} \quad h \leqslant z \leqslant d. \tag{2.17}$$

$\partial\psi_m/\partial r$ may be expanded over the whole interval $0 \leqslant z \leqslant d$ as

$$\partial\psi_m/\partial r = \sum_\alpha \mathscr{F}_{m\alpha} Z_\alpha(z), \tag{2.18}$$

where

$$\mathscr{F}_{m\alpha} = \frac{1}{d}\int_0^h f_m(z)\, Z_\alpha(z)\, dz \tag{2.19}$$

and $\sum_\alpha$ denotes summation over α including $\alpha = -ik$, with corresponding suffix k, as the first term as well as all the positive real roots of (2.13).

We may now write down the appropriate expansion of ψ_m in $r \geqslant a$ and $r \leqslant a$ in terms of $f_m(z)$. In $r \geqslant a$

$$\psi_m(r,z) = \left\{J_m(kr) - \frac{J_m'(ka)}{H_m'(ka)}\right\} H_m(kr)\frac{Z_k(z)}{Z_k'(d)} + \sum_\alpha \mathscr{F}_{m\alpha}\frac{K_m(\alpha r)}{\alpha K_m'(\alpha a)} Z_\alpha(z), \tag{2.20}$$

where $H_m = J_m + iY_m$ is the Hankel function of the first kind (the usual superscript 1 may be omitted without ambiguity) and note that

$$K_m(-ikr) = \tfrac{1}{2}\pi i^{m+1} H_m(kr). \tag{2.21}$$

Hankel functions of the second kind do not enter (2.20) as the scattered wave must satisfy a radiation condition, and the Bessel function I_m is omitted as it is unbounded at infinity. In (2.20) the first term represents the incoming wave, the second term represents the wave that would be scattered if the cylinder extended to the bottom, and the infinite sum represents a further scattered wave (from $\alpha = -ik$) together with a set of modes increasingly localized near the cylinder.

In $r \leqslant a$

$$\psi_m(r,z) = \sum_\alpha \mathscr{F}_{m\alpha}\frac{I_m(\alpha r)}{I_m'(\alpha a)} Z_\alpha(z), \tag{2.22}$$

where the first term in the sum has $\alpha = -ik$ for which

$$I_m(-ikr) = (-i)^m J_m(kr). \tag{2.23}$$

Other modes are excluded by the requirement that ψ_m be bounded at the origin.

Now the pressure, hence the azimuthal component of displacement and hence ψ_m must be continuous at $r = a$ for $0 \leqslant z < h$, so that

$$\left[J_m(ka) - \frac{J_m'(ka)}{H_m'(ka)} H_m(ka)\right]\frac{Z_k(z)}{Z_k'(d)} = \sum_\alpha \mathscr{F}_{m\alpha}\left[\frac{I_m(\alpha a)}{\alpha I_m'(\alpha a)} - \frac{K_m(\alpha a)}{\alpha K_m'(\alpha a)}\right] Z_\alpha(z). \tag{2.24}$$

We now use the formulae

$$J_m(ka)\,H_m'(ka) - J_m'(ka)\,H_m(ka) = 2i(\pi ka)^{-1}, \tag{2.25}$$

$$I_m(\alpha a)\,K_m'(\alpha a) - I_m'(\alpha a)\,K_m(\alpha a) = (\alpha a)^{-1}, \tag{2.26}$$

and define

$$F_m = 2i[\pi k a^2 H_m'(ka)\,Z_k'(d)]^{-1}, \tag{2.27}$$

$$R_\alpha = -[\alpha^2 a^2 I_m'(\alpha a)\,K_m'(\alpha a)]^{-1}. \tag{2.28}$$

Equation (2.24) then becomes

$$F_m Z_k(z) = \sum_\alpha R_\alpha \mathscr{F}_{m\alpha} Z_\alpha(z). \tag{2.29}$$

3. Solution

Equation (2.29) is valid over the interval $0 \leqslant z < h$, and over $h \leqslant z \leqslant d$ we have

$$0 = \sum_\alpha \mathscr{F}_{m\alpha} Z_\alpha(z). \tag{3.1}$$

Multiplying (2.29) and (3.1) by $Z_\beta(z)$, integrating each over the region of validity, adding and dividing by d, we obtain

$$F_m C_\beta = \sum_\alpha E_{\beta\alpha}\mathscr{F}_{m\alpha}, \tag{3.2}$$

in which
$$C_\beta = \frac{1}{d}\int_0^h Z_k(z)\,Z_\beta(z)\,dz, \tag{3.3}$$
$$E_{\beta\alpha} = (R_\alpha - 1)\frac{1}{d}\int_0^h Z_\alpha(z)\,Z_\beta(z)\,dz + \delta_{\beta\alpha}. \tag{3.4}$$

Equation (3.2) is an infinite set of simultaneous linear equations for $\mathscr{F}_{m\alpha}$. The solution of a finite number of these may not converge as rapidly as that obtained using other formulations, but the present approach has certain advantages. This will be discussed further in § 7.

Define
$$D_{\beta\alpha} = \frac{1}{d}\int_0^h Z_\alpha(z)\,Z_\beta(z)\,dz, \tag{3.5}$$

then the actual values of **C**, **D** are
$$C_k = \tfrac{1}{2}(h/d)N_k^{-1}\left(1+\frac{\sinh 2kh}{2kh}\right), \tag{3.6}$$
$$C_\beta = (N_k N_\beta)^{-\frac{1}{2}}\,(\beta^2d^2+k^2d^2)^{-1}\,(\beta d\sin\beta h\cosh kh + kd\cos\beta h\sinh kh), \tag{3.7}$$
$$D_{k\alpha} = D_{\alpha k} = C_\alpha, \tag{3.8}$$
$$D_{\beta\alpha} = (N_\alpha N_\beta)^{-\frac{1}{2}}\,(\alpha^2d^2-\beta^2d^2)^{-1}\,(\alpha d\sin\alpha h\cos\beta h - \beta d\cos\alpha h\sin\beta h) \quad \text{for} \quad \alpha \neq \beta, \tag{3.9}$$
$$D_{\beta\beta} = \tfrac{1}{2}(h/d)N_\beta^{-1}\left(1+\frac{\sin 2\beta h}{2\beta h}\right). \tag{3.10}$$

C and **D** are real. R_α is real unless $\alpha = -ik$, for which
$$R_k = -[\tfrac{1}{2}i\pi k^2a^2J'_m(ka)\,H'_m(ka)]^{-1}. \tag{3.11}$$

Thus (3.2) may be written
$$F_m C_\beta = \sum_\alpha (E^{(r)}_{\beta\alpha} + R_k C_\beta\,\delta_{k\alpha})\,\mathscr{F}_{m\alpha}, \tag{3.12}$$

where $E^{(r)}_{\beta\alpha}$ is real, and defined by
$$E^{(r)}_{\beta\alpha} = (R_\alpha - 1)\,D_{\beta\alpha} - R_k C_\beta\,\delta_{k\alpha} + \delta_{\beta\alpha}. \tag{3.13}$$

Assume that $E^{(r)}_{\beta\alpha}$ is non-singular and suppose that Φ_α is the solution of
$$C_\beta = \sum_\alpha E^{(r)}_{\beta\alpha}\,\Phi_\alpha. \tag{3.14}$$

The solution of (3.12) is then easily seen to be
$$\mathscr{F}_{m\alpha} = \frac{F_m\,\Phi_\alpha}{1+R_k\,\Phi_k}. \tag{3.15}$$

But Φ_α is real, so (3.15) gives us the important result that *the phase of* $\mathscr{F}_{m\alpha}$ *is independent of* α, *i.e. the phase of* $f_m(z)$ *is independent of* z.

We may also use (3.15) to derive a useful expression for $\mathscr{F}_{mk}$, which is the most important quantity in the solution. Using the definitions of F_m, R_k in (2.27), (3.11) we may write $\mathscr{F}_{mk}$ from (3.15) as
$$\mathscr{F}_{mk} = \frac{kJ'_m(ka)}{Z'_k(d)}\,[1+R_k^{-1}\,\Phi_k^{-1}]^{-1}. \tag{3.16}$$

But multiplying (2.29) by $f_m(z)$ and integrating over $[0, h]$ we have

$$F_m \mathscr{F}_{mk} = \sum_\alpha R_\alpha \mathscr{F}^2_{m\alpha} \tag{3.17}$$

or, using (3.15)

$$\Phi_k = \sum_\alpha{}' R_\alpha \Phi^2_\alpha, \tag{3.18}$$

where $\sum_\alpha'$ denotes summation over only the real roots α, i.e. $\alpha = -ik$ is excluded. Thus (3.16) may be written

$$\mathscr{F}_{mk} = \frac{kJ'_m(ka)}{Z'_k(d)}\left[1 + \tfrac{1}{2}i\pi k^2a^2J'_m(ka)\,H'_m(ka)\sum_\alpha{}' \frac{\Phi^2_\alpha/\Phi^2_k}{\alpha^2a^2I'_m(\alpha a)\,K'_m(\alpha a)}\right]^{-1} \tag{3.19}$$

$$= \frac{kJ'_m(ka)}{Z'_k(d)}\left[1 + \tfrac{1}{2}i\pi k^2a^2J'_m(ka)\,H'_m(ka)\sum_\alpha{}' \frac{\mathscr{F}^2_{m\alpha}/\mathscr{F}^2_{mk}}{\alpha^2a^2I'_m(\alpha a)\,K'_m(\alpha a)}\right]^{-1}. \tag{3.20}$$

In § 6 we shall see that the right-hand side of (3.20) is stationary with respect to small variations of $f_m(z)$ about a constant times the exact solution, and so may be used to calculate a good approximation to $\mathscr{F}_{mk}$. However, in § 4 we use (3.19) to deduce quite a lot about the behaviour of the solution, especially close to a value of ka for which J'_m vanishes.

4. Resonance

Using (2.7) and (2.22) the free surface elevation of (2.5) has

$$\chi_m(r) = A_m J_m(kr) + \sum_\alpha{}' \mathscr{F}_{m\alpha} \frac{I_m(\alpha r)}{\alpha I'_m(\alpha a)} Z'_\alpha(d) \tag{4.1}$$

in $r \leqslant a$, where

$$A_m = \mathscr{F}_{mk} \frac{Z'_k(d)}{kJ'_m(ka)}. \tag{4.2}$$

The solution for $r \geqslant a$ may be written

$$\chi_m(r) = J_m(kr) + B_m H_m(kr) + \sum_\alpha{}' \mathscr{F}_{m\alpha} \frac{K_m(\alpha r)}{\alpha K'_m(\alpha a)} Z'_\alpha(d), \tag{4.3}$$

where

$$B_m = (A_m - 1)\frac{J'_m(ka)}{H'_m(ka)}. \tag{4.4}$$

The first term on the right-hand side of (4.1) is the most important contribution to the wave motion inside the cylinder, the other terms describe waves which are generally confined near $r = a$. From (3.19) we may write

$$A_m = \left[1 + \tfrac{1}{2}i\pi k^2a^2J'_m(ka)\,H'_m(ka)\sum_\alpha{}' \frac{\Phi^2_\alpha/\Phi^2_k}{\alpha^2a^2I'_m(\alpha a)\,K'_m(\alpha a)}\right]^{-1}. \tag{4.5}$$

The first thing we notice about this is that at a zero of $J'_m(ka)$, $A_m = 1$ and $\mathscr{F}_{m\alpha} = 0$ (as $R_k \to \infty$ in (3.15)). Thus (4.1) and (4.3) become

$$\chi_m(r) = J_m(kr) \quad \text{for all} \quad r. \tag{4.6}$$

This solution is only to be expected; if $J'_m(ka) = 0$ no scattered waves or waves localized near $r = a$ are required to satisfy the boundary conditions and matching conditions for ψ_m provided that $\chi_m(r) = J_m(kr)$ inside the cylinder. We might expect this solution to be the resonant one, in the sense of giving the largest value of $|A_m|$. However, this is not the case, as can be seen by examining (4.5) further for ka close to a zero of J'_m.

Suppose that

$$ka = j'_m + p, \tag{4.7}$$

where

$$J'_m(j'_m) = 0, \tag{4.8}$$

then

$$A_m \doteqdot \left[1 + \tfrac{1}{2} i\pi j'^2_m J''_m(j'_m)\, p [J''_m(j'_m)\, p + i Y'_m(j'_m)] \sum_\alpha{}' \frac{\Phi^2_\alpha/\Phi^2_k}{\alpha^2 a^2 I'_m(\alpha a)\, K'_m(\alpha a)}\right]^{-1}. \tag{4.9}$$

There is nothing special about j'_m as far as the coefficients in (3.14) are concerned, thus the infinite sum in (4.9) is a smooth finite function of ka and may be treated as constant in the neighbourhood of j'_m. Denote

$$\lambda = \tfrac{1}{2}\pi j'^2_m J''_m(j'_m)\, Y'_m(j'_m) \left(\sum_\alpha{}' \frac{\Phi^2_\alpha/\Phi^2_k}{\alpha^2 a^2 I'_m(\alpha a)\, K'_m(\alpha a)}\right)_{ka=j'_m} \tag{4.10}$$

$$\mu = -\frac{J''_m(j'_m)}{Y'_m(j'_m)}. \tag{4.11}$$

In (4.10) $I'_m(\alpha a)\, K'_m(\alpha a) < 0$ (in fact $I'_m(\alpha a)\, K'_m(\alpha a) \sim -(2\alpha a)^{-1}$ for large $\alpha a/m$). Also $J''_m(j'_m)$ and $Y'_m(j'_m)$ are of opposite sign. Thus λ and μ are both positive, and (4.9), (4.4) give

$$A_m = [(1-\lambda p) - i\lambda\mu p^2]^{-1}, \tag{4.12}$$

$$B_m = i\lambda\mu p^2[(1-\lambda p) - i\lambda\mu p^2]^{-1}, \tag{4.13}$$

for which

$$|A_m| = \mathscr{A}_m(p) = [(1-\lambda p)^2 + \lambda^2\mu^2 p^4]^{-\frac{1}{2}}, \tag{4.14}$$

$$|B_m| = \mathscr{B}_m(p) = \lambda\mu p^2[(1-\lambda p)^2 + \lambda^2\mu^2 p^4]^{-\frac{1}{2}}. \tag{4.15}$$

For large λ (as will generally be the case) $\mathscr{A}_m(p)$ has a maximum of

$$\frac{\lambda}{\mu}\left[1 + 2\frac{\mu^2}{\lambda^2} + O\left(\frac{\mu^4}{\lambda^4}\right)\right] \quad \text{at} \quad p = \frac{1}{\lambda}\left[1 - 2\frac{\mu^2}{\lambda^2} + O\left(\frac{\mu^4}{\lambda^4}\right)\right].$$

$\mathscr{B}_m(p)$ has a minimum of 0 at $p = 0$, a maximum of 1 at $p = 1/\lambda$ and a minimum of

$$4\frac{\mu}{\lambda}\left(1 + 16\frac{\mu^2}{\lambda^2}\right)^{-\frac{1}{2}} \quad \text{at} \quad p = \frac{2}{\lambda}.$$

Figure 2 shows $\mathscr{A}_m(p)$, $\mathscr{B}_m(p)$ for $\lambda = 5$, $\mu = 1$. For comparison with $\mathscr{B}_m(p)$ figure 2 also shows the amplitude, $|\mu p|(1+\mu^2 p^2)^{-\frac{1}{2}}$, of the wave that would be scattered by a cylinder extending to the bottom.

This behaviour at resonance is not unusual. Perhaps one can understand it as follows: At $p = 0$ the frequency of the incident wave is perfectly tuned to the natural frequency of a mode inside the cylinder, but the response is only $\mathscr{A}_m = 1$ because $r = a$ is a turning point and so the coupling is poor. For $p > 0$ the coupling increases more rapidly than the system is detuned, and so the response reaches a maximum at some small positive value of p. This does not happen for $p < 0$ due to a mismatch in the phases of the incident wave and oscillation inside the cylinder.

The sharpness of a given resonance will depend largely on the value of $k(d-h)$. The deeper the cylinder extends compared with the wavelength of the incident wave, the sharper will be the resonance. This will be discussed further in § 7 in terms of the numerical results obtained there.

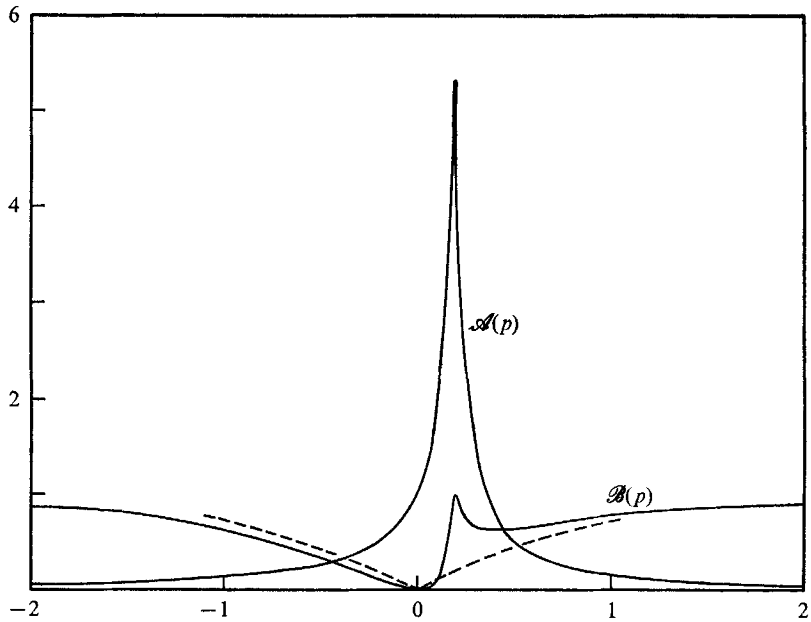

FIGURE 2. Behaviour near resonance ($\lambda = 5$, $\mu = 1$). $\mathscr{A}(p)$ is the amplitude of the harbour oscillation, $\mathscr{B}(p)$ the amplitude of the scattered wave. The dashed line gives the amplitude of the scattered wave for a cylinder extending to the bottom.

The above analysis does not apply if $m = 0$ and there is a resonance for small ka. In this case define $\lambda_0 = -\frac{1}{2}\{\sum_\alpha'(\Phi_\alpha^2/\Phi_k^2)[\alpha^2a^2I_m'(\alpha a)\,K_m'(\alpha a)]^{-1}\}_{ka=0}$ and it is easily seen that

$$A_0 \doteqdot [1-\lambda_0(ka)^2 - i\lambda_0\tfrac{1}{4}\pi(ka)^4]^{-1} \tag{4.16}$$

which is of the form (4.12) with $\mu = \frac{1}{4}\pi$ and p replaced by $(ka)^2$. For this to be a significant resonance would require $(d-h) \gg a$. This resonance thus corresponds to the manometric oscillations in an open pipe discussed by Isaacs & Wiegel (1949). The present theory could be used to predict the amplitude of such oscillations, but in doing so one would have to include the motion described by the infinite series of modified Bessel functions. These would no longer be localized near $r = a$ as we would not have large αa.

5. Response to a continuous spectrum

In practice we would require the response of the harbour to a continuous spectrum of incident waves, rather than the plane wave treated so far. In considering the contribution from a sharp resonance at $ka = j_m'$ to the mean-square surface displacement inside the cylinder we need to know the value of the integral

$$I = \int_{-\infty}^{\infty} \frac{dp}{(1-\lambda p)^2+\lambda^2\mu^2p^4}. \tag{5.1}$$

This may be evaluated by contour integration as

$$I = \frac{8\pi\mu}{\lambda^2} \frac{\left\{\frac{1}{2}\left[1+\left(1+\frac{16\mu^2}{\lambda^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}}{\left(1+\frac{16\mu^2}{\lambda^2}\right)^{\frac{1}{2}}\left[\left(1+\frac{16\mu^2}{\lambda^2}\right)^{\frac{1}{2}}-1\right]} \tag{5.2}$$

$$= \frac{\pi}{\mu}\left[1-\frac{\mu^2}{\lambda^2}+O\left(\frac{\mu^4}{\lambda^4}\right)\right]. \tag{5.3}$$

m	j'_m	$\mu = -\dfrac{J''_m(j'_m)}{Y'_m(j'_m)}$	$\left(\dfrac{\pi}{j'_m\mu}\right)^{\frac{1}{2}} \epsilon_m$
0	3·83	0·976	0·916
	7·02	0·993	0·746
1	1·84	0·690	3·14
	5·33	0·968	1·56
	8·54	0·988	1·22
2	3·05	0·649	2·52
	6·71	0·943	1·41
	9·97	0·975	1·14
3	4·20	0·610	2·21
	8·02	0·918	1·30
4	5·32	0·579	2·02
	9·28	0·894	1·23
5	6·42	0·554	1·88
6	7·50	0·532	1·77
7	8·58	0·514	1·69
8	9·65	0·498	1·62

TABLE 1. Zeros of $J'_m(ka)$ less than 10, and associated quantities which describe the resonance of the harbour near $ka = j'_m$ (see §§ 4, 5)

If the incident waves have a frequency spectrum $S(\sigma)$, then the spectrum as a function of ka is $(1/a)\, S(\sigma)\, d\sigma/dk$. Thus the contribution to the root-mean-square elevation at (r, θ) inside the cylinder due to the zero j'_m is given approximately by $\bar{A}_m J_m(j'_m r/a) \cos m\theta$ where

$$\bar{A}_m = \left[\sigma S(\sigma)\frac{k}{\sigma}\frac{d\sigma}{dk}\right]^{\frac{1}{2}} \left(\frac{\pi}{j'_m\mu}\right)^{\frac{1}{2}} \epsilon_m. \tag{5.4}$$

This assumes a uni-directional spectrum. For a directional spectrum the dependence on θ will be different but the order of magnitude much the same. Values of j'_m, μ and $(\pi/j'_m\mu)^{\frac{1}{2}}\,\epsilon_m$ for all $j'_m < 10$ are given in table 1.

Thus for sharp resonances the response of the harbour to a continuous spectrum is independent of the detailed solution of the problem, but is given by the remarkably simple expression (5.4). However sharp the resonance the r.m.s response is still finite. A similar effect for long waves trapped by circular sea-mounts has been found by Longuet-Higgins (1967), and also occurs for the excitation of harbour oscillations through the harbour entrance (see the appendix).

Of course any departure from the idealized model treated here will alter this conclusion. The effect of a harbour entrance in particular will be discussed in § 8.

6. Integral equation and variational approximation

Using the definition (2.19) of $\mathscr{F}_{m\alpha}$ in terms of $f_m(z)$, (2.29) may be written as the integral equation

$$F_m Z_k(z) = \int_0^h g_m(z, \zeta) f_m(\zeta)\, d\zeta, \tag{6.1}$$

where

$$g_m(z, \zeta) = \frac{1}{d} \sum_\alpha R_\alpha Z_\alpha(z) Z_\alpha(\zeta). \tag{6.2}$$

Equation (6.1) is a non-singular Fredholm integral equation of the first kind with a symmetric kernel.

A variational approximation for $\mathscr{F}_{mk}$ equivalent to that used by Miles & Gilbert (1968) may readily be derived. Multiply (6.1) by $f_m(z)$, integrate over $0 \leqslant z \leqslant h$ and divide by $d\mathscr{F}^2_{mk}$ to obtain

$$\frac{F_m}{\mathscr{F}_{mk}} = \frac{d \int_0^h \int_0^h f_m(z)\, g_m(z, \zeta) f_m(\zeta)\, d\zeta\, dz}{\left[\int_0^h f_m(z)\, Z_k(z)\, dz\right]^2}. \tag{6.3}$$

As remarked by Miles & Gilbert (1968) for an equation of exactly the same form, the standard variational procedure leads to the result that the right-hand side of (6.3) is stationary with respect to small variations of $f_m(z)$ about a constant factor times the exact solution of (6.1) (since (6.3) is also independent of the scale of $f_m(z)$). In other words, we may estimate $\mathscr{F}_{mk}$ to $O(\epsilon^2)$ by evaluating the right-hand side of (6.3) for a trial function which is within $O(\epsilon)$ of a constant factor times the true solution, where ϵ is a small number. Moreover, it was proved in § 3 that the true solution has a phase independent of z, so that it is adequate to use a real trial function.

With a trial function $f^*_m(z)$ and associated coefficients $\mathscr{F}^*_{m\alpha}$ defined by (2.19), (6.3) leads to (cf. (3.20))

$$\mathscr{F}_{mk} = \frac{kJ'_m(ka)}{Z'_k(d)} \left[1 + \tfrac{1}{2} i\pi k^2 a^2 J'_m(ka)\, H'_m(ka) \sum_\alpha \frac{\mathscr{F}^{*2}_{m\alpha} / \mathscr{F}^{*2}_{mk}}{\alpha^2 a^2 I'_m(\alpha a)\, K'_m(\alpha a)}\right]^{-1}. \tag{6.4}$$

Miles & Gilbert derived all their results using the horizontal particle displacement of the incoming wave as the trial function, i.e. they took $f^*_m(z) = Z_k(z)$ whence $\mathscr{F}^*_{m\alpha} = C_\alpha$ of (3.3). The accuracy of this for the present problem will be discussed in § 7.

7. Numerical methods and results

We are primarily interested in $\mathscr{F}_{mk}$, as this gives the amplitude of the oscillation within the cylinder except near $r = a$. An approximate value of $\mathscr{F}_{mk}$ may be obtained from (6.4) if a suitable trial function is chosen. Miles (1967) has shown that the use of the incoming wave displacement as a trial function gives results for the transmission and reflexion of surface waves at a shelf which are in remarkably good agreement with the detailed computations of Newman (1965). In the present problem the singularity at $r = a$, $z = h$ is more severe, though, and it turns out that the incoming wave displacement is not always an adequate trial function (see later). Of course one could include the singularity in the trial function; excellent results could probably be obtained by using a trial function given by the sum of the incoming wave displacement and a factor times the displacement $(h-z)^{-\frac{1}{2}}$ associated with the singularity, the factor being determined from the stationarity of (6.3).

However, the numerical solution of a set of equations such as (3.2), or at any rate a large number of them, is rapid. Convergence may well be slow as the number N of equations taken is increased, but one can extrapolate to infinity and know the accuracy of one's solution, whereas it is difficult to assess the accuracy of a variational approximation. Moreover, solving the set of simultaneous equations gives one all the $\mathscr{F}_{m\alpha}$, which could be used to calculate the force on the cylinder or the peripheral disturbance (though these calculations will not be performed here).

We must now decide how to represent the exact solution and how to solve for it. It seems simpler to solve directly for $\mathscr{F}_{m\alpha}$, or rather Φ_α, than to expand $f_m(z)$ in a set of functions complete over $[0, h]$ and solve (2.29) for the coefficients of these. We could solve the first N equations of (3.2) or (3.14), but a more satisfactory technique is probably to minimize the error of (2.29), (3.1) in least squares, as suggested by Sommerfeld (1949, ch. 1, §6C) for a similar problem. First we use (3.15) to obtain the real equations

$$\sum_1^\infty \Phi_\alpha Z_\alpha = 0 \quad \text{for} \quad h \leqslant z \leqslant d, \tag{7.1}$$

$$\sum_2^\infty R_\alpha \Phi_\alpha Z_\alpha = Z_k \quad \text{for} \quad 0 \leqslant z < h, \tag{7.2}$$

where the summation is over α and (7.2) omits $\alpha = -ik$. We now minimize

$$\int_h^d \left(\sum_1^N \Phi_\alpha Z_\alpha\right)^2 dz + \int_0^h \left(\sum_2^N R_\alpha \Phi_\alpha Z_\alpha - Z_k\right)^2 dz. \tag{7.3}$$

We could take a weighting function inside each of these integrals and we could weight each integral differently. The present approach is the simplest, not necessarily the best. Making (7.3) stationary gives

$$\sum_1^N \mathscr{E}_{\beta\alpha} \Phi_\alpha = R_\beta C_\beta - \delta_{k\beta} R_k C_k, \tag{7.4}$$

where

$$\left.\begin{aligned} \mathscr{E}_{kk} &= 1 - C_k, \quad \mathscr{E}_{k\alpha} = \mathscr{E}_{\alpha k} = -C_\alpha \quad \text{for} \quad \alpha \neq k, \\ \mathscr{E}_{\beta\alpha} &= \delta_{\beta\alpha} + (R_\beta R_\alpha - 1) D_{\beta\alpha} \quad \text{for} \quad \alpha \neq k,\ \beta \neq k. \end{aligned}\right\} \tag{7.5}$$

Equation (7.4) may now be solved numerically for any choice of N. The solution is slow to converge because of the singularity $f_m(z) \propto (h-z)^{-\frac{1}{2}}$ near $z = h$. However Φ_k is found to be almost linearly dependent on $1/N$ for large N. For representative values of the parameters within the results shown here (7.4) was solved for $N = 10, 20, 30, 40, 50, 60, 70, 80$ and the linear extrapolation to $1/N = 0$ using $N = 20$ and $N = 40$ compared with a smooth extrapolation using all the values of N. The results generally agreed very well, the greatest discrepancy being less than 3 %, and that in a situation where the answer for $N = 40$ was inaccurate by 15 %. Thus linear extrapolation (in N^{-1}) from the solutions for $N = 20$ and $N = 40$ has been used to derive all the results presented here. Admittedly the values of d/a taken here are not large; for larger d/a and certain values of h/a, ka one would need to take larger values of N.

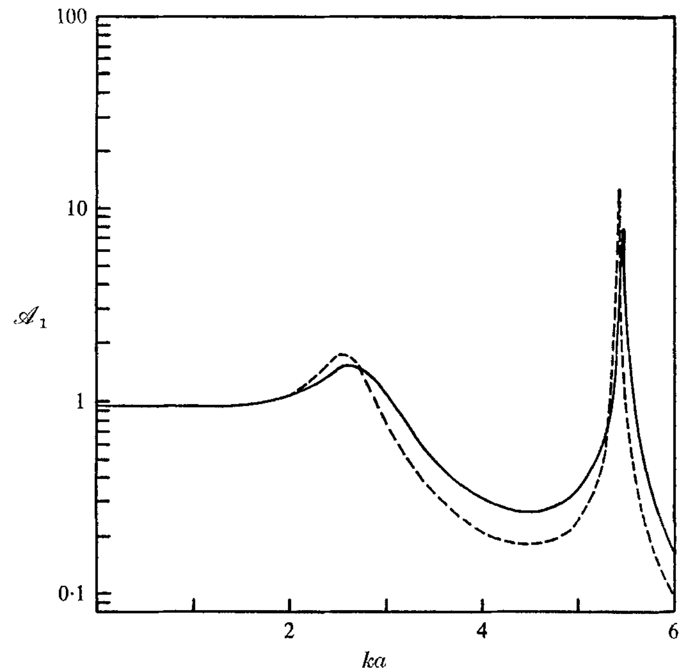

FIGURE 3. Comparison of variational approximation (– – –) with exact solution (——) for $m = 1$, $d/a = 0{\cdot}6$, $h/a = 0{\cdot}4$.

Having solved for the $\mathscr{F}_{1\alpha}$ for a given N one might hope to improve the accuracy of $\mathscr{F}_{1k}$ by feeding the detailed solution back into the variational approximation by defining a trial function

$$f_m^*(z) = \sum_1^N \mathscr{F}_{m\alpha} Z_\alpha(z) \quad (0 \leqslant z < h), \tag{7.6}$$

whence

$$\mathscr{F}^*_{m\beta} = \sum_1^N D_{\beta\alpha} \mathscr{F}_{m\alpha}. \tag{7.7}$$

In practice, though, this does not produce a significant increase in accuracy, again because of the singularity at $z = h$. (7.6) must be within $O(1/N)$ of the true solution except within $O(1/N)$ of $z = h$. The inaccuracy in the variational approximation from this small region will be formally $O(1/N^2)$, but actually

$$O\left[\left(\int_0^{1/N} \zeta^{-\frac{1}{2}} d\zeta\right)^2\right] = O(1/N).$$

Figure 3 shows $\mathscr{A}_1 (= |A_1|$, see (4.2)) as a function of ka for $d/a = 0{\cdot}6$, $h/a = 0{\cdot}4$. The full line shows the exact solution (i.e. the extrapolation from $N = 20, 40$) and the dotted line shows the variational approximation using $f_1^*(z) = Z_k(z)$. We see that the variational approximation is extremely good for small ka, but deteriorates as ka increases and is out by a factor of nearly 2 at

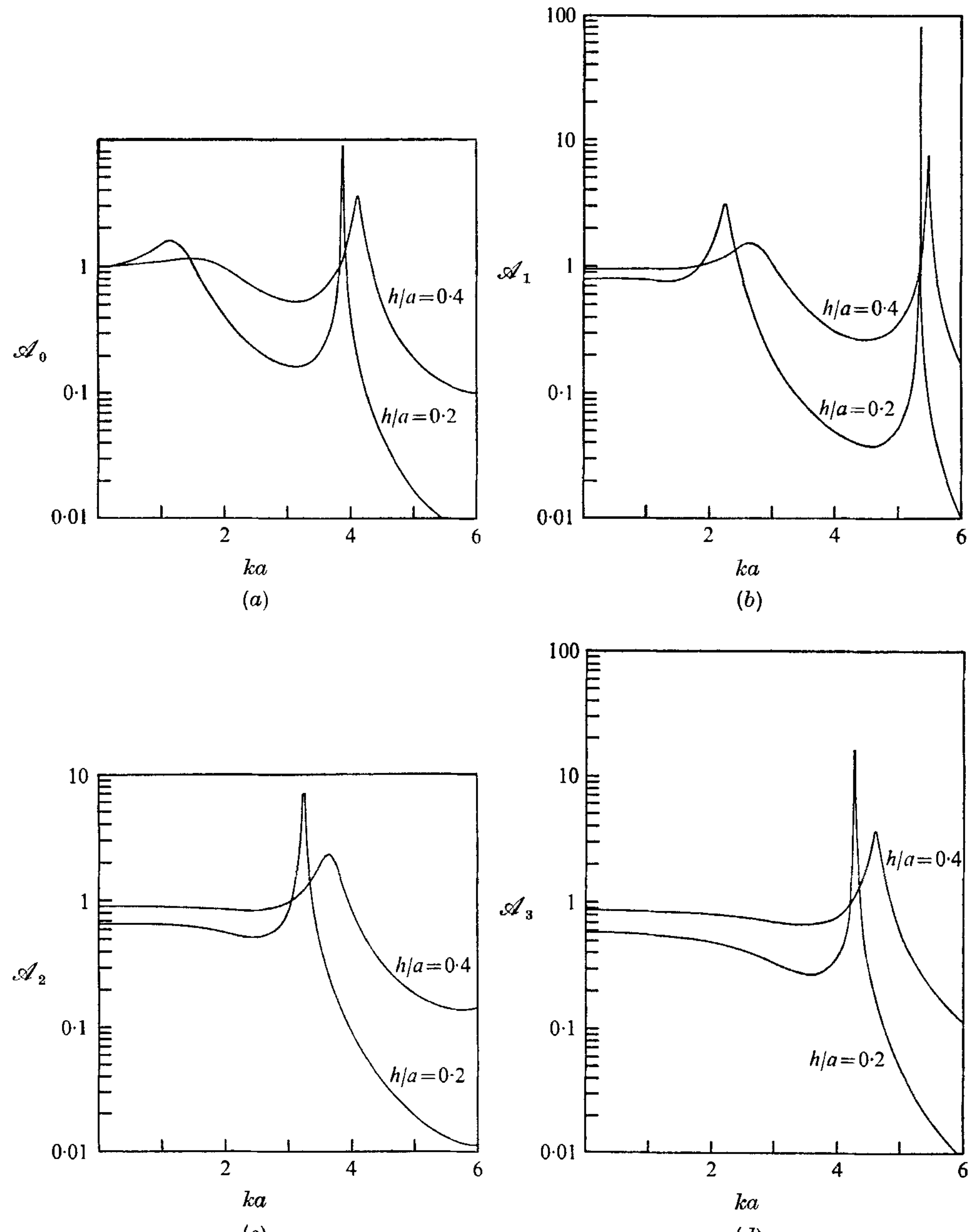

FIGURE 4. Amplitude of harbour oscillation. $\mathscr{A}_m$ for $d/a = 0{\cdot}6$, $h/a = 0{\cdot}4$, $0{\cdot}2$. (*a*) $m = 0$, (*b*) $m = 1$, (*c*) $m = 2$, (*d*) $m = 3$.

$ka = 6$. This may be understood in terms of the effect of the singularity at $z = h$; for large values of ka, $Z_k(z)$ decreases rapidly with depth, and so the singularity at $z = h$ will drastically alter $f_1(z)$ in the only part of $[0, h]$ where Z_k is not negligible compared with its value at $z = h$.

Figure 4 shows $\mathscr{A}_m$ for $d/a = 0{\cdot}6$, $h/a = 0{\cdot}4$, $0{\cdot}2$, $m = 0, 1, 2, 3$ and illustrates the effect of increasing $d-h$. For a given ka the value of $\mathscr{A}_m$ as a function of d/a and h/a is determined mainly by $(d-h)/a$ is illustrated in figure 5 which shows $\mathscr{A}_1$ for $d/a = 0{\cdot}4$, $0{\cdot}8$ and $(d-h)/a = 0{\cdot}2$.

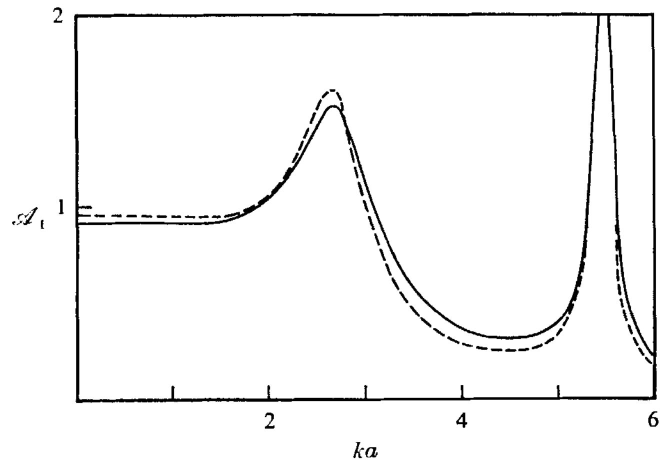

FIGURE 5. Dependence of $\mathscr{A}_1$ on d/a for fixed $(d-h)/a$. ——, $d/a = 0{\cdot}4$, $h/a = 0{\cdot}2$; ---, $d/a = 0{\cdot}8$, $h/a = 0{\cdot}6$.

There is a slight tendency for the resonances to become sharper as d increases, but the effect is small; for fixed $(d-h)/a$ the value of $\mathscr{A}_m$ at a given ka changes very little as kd increases beyond about 2.

A rule of thumb emerging from the results is that apart from the resonances the amplitude of the motion inside the cylinder is less than 10 % of the amplitude of the incoming waves provided that the cylinder is immersed to a depth greater than about a quarter of the wavelength. This is comparable with Ursell's (1947) results for the transmission of waves past a vertical barrier.

8. Applications

Any application of the foregoing theory and results to a design for a particular artificial island must take into account the following differences between the model treated here and reality:

(i) the harbour will probably not be circular;
(ii) the walls of the harbour will not be infinitely thin;
(iii) the harbour must have an entrance.

We ignore (i) as any harbour will have free modes of oscillation and the response of these will be much the same as for the circular cylinder considered. Finite wall thickness probably reduces the response of the harbour, but only by a small amount. The effect of the harbour entrance is less obvious. The harbour entrance is itself associated with generation of the modes inside the harbour and also with radiative damping of these modes. At a resonant frequency it seems quite possible that a mode generated through the bottom could leak out through the harbour entrance before it could reach a large amplitude, or vice versa.

Quantitatively one might describe a given mode inside the harbour as an oscillator which is both being forced and decaying in two different ways and is thus governed by the equation

$$\ddot{x}+2(Q_e^{-1}+Q_b^{-1})\,p\dot{x}+p^2x = \operatorname{Re}\,[(F_e+F_b)\,p^2\,e^{i\omega t}]. \tag{8.1}$$

x is the amplitude of the oscillator and p its natural frequency. Q_e, Q_b are the Q's of the oscillator associated with the harbour entrance and bottom respectively and F_e, F_b denote the complex amplitudes of the forcing functions for an incident wave of unit amplitude. Of course Q_e, Q_b, F_e, F_b may all be functions of ω. The solution of (8.1) is

$$x = \operatorname{Re}\,[(F_e+F_b)\,Z^{-1}\,e^{i\omega t}], \tag{8.2}$$

where

$$Z = 1-\frac{\omega^2}{p^2}+\frac{i\omega}{p}\,(Q_e^{-1}+Q_b^{-1}). \tag{8.3}$$

Away from resonance $x \doteqdot \operatorname{Re}\,[(F_e+F_b)\,(1-(\omega^2/p^2))^{-1}\,e^{i\omega t}]$ which is the sum of the responses of the oscillator to each forcing mechanism separately. The most important quantity associated with the resonance is the 'power transfer factor'

$$P = \int_0^\infty |F_e+F_b|^2\,|Z|^{-2}\,d\omega, \tag{8.4}$$

which, when multiplied by the spectrum of the incident wave at $\omega = p$, gives the mean square amplitude of the oscillator. In (8.4) Q_e, Q_b take their values at $\omega = p$. Then

$$P = \tfrac{1}{2}\pi p\,|F_e+F_b|^2\,(Q_e^{-1}+Q_b^{-1})^{-1}. \tag{8.5}$$

We must now relate F_e to Q_e and F_b to Q_b. At resonance (4.12) resembles the response of an oscillator with $Q_b = \lambda^2\mu^{-1}$ and a constant forcing function $F_b = \lambda^{-1}$. Thus, taking $\mu = 1$, $F_b = Q_b^{-\frac{1}{2}}$. As shown in § 5 this on its own gives a power transfer factor $O(1)$ independent of Q_b. The excitation through the harbour entrance may similarly be described by a forcing function $F_e = \nu Q_e^{-\frac{1}{2}}$ where ν is a constant (see the appendix). The entrance on its own has a power transfer factor $O(\nu^2)$. Assuming F_e and F_b to be in phase the total power transfer factor may be written

$$P \propto (\nu Q_e^{-\frac{1}{2}}+Q_b^{-\frac{1}{2}})^2\,(Q_e^{-1}+Q_b^{-1})^{-1}. \tag{8.6}$$

Thus if Q_b/Q_e is large enough, the effect of resonant excitation through the harbour bottom is negligible; the excited wave leaks out through the harbour entrance instead of building up to a large amplitude.

It is a pleasure to thank Professors John W. Miles and Freeman Gilbert for fruitful discussions on this and related problems. This work was supported by the National Science Foundation.

Appendix. The response of a harbour to excitation through its entrance

Miles & Munk (1961) discussed the excitation of a harbour oscillation by regarding it as a forced oscillator with a given Q and a forcing function of amplitude one. Thus they found a peak response at resonance of Q and a power transfer factor of $O(Q)$ (i.e. the mean square response of the harbour to a continuous spectrum is $O(Q)$ times the spectral density at the resonant frequency). Narrowing the entrance to a harbour increases its Q and so they were led to the paradoxical result that decreasing the width of the entrance to a harbour decreases its protection from the external waves.

This is incorrect, however, due to the erroneous assumption that the appropriate forcing function has amplitude one. In fact the detailed calculations of Miles & Munk for a rectangular harbour of length d, width b, and entrance width a indicate a peak response of $O(Q^{\frac{1}{2}})$ with corresponding power transfer factor $O(1)$. In the following interpretation of their solution the second number in brackets refers to an equation number of Miles & Munk.

They show that the disturbance in the harbour may be described by

$$\frac{\zeta(x,y)}{\zeta_i(0,0)} = \frac{2}{D(k)}\int_{-\frac{1}{2}a}^{\frac{1}{2}a} G(x,y,\eta)\,\phi(\eta)\,d\eta, \qquad \text{(A 1, 38)}$$

where ζ_i is the amplitude of the incident wave. $G(x,y,\eta)$ is a Green's function given by

$$G(x,y,\eta) = -\frac{\cos[k(x+d)]}{kb\sin kd} + \begin{matrix}\text{an infinite sum of modes trapped}\\ \text{near the entrance.}\end{matrix} \qquad \text{(A 2, 46)}$$

$\phi(\eta)$ describes the shape of the surface elevation in the harbour entrance and is normalized so that

$$\int_{-\frac{1}{2}a}^{\frac{1}{2}a} \zeta_i(0,y)\,\phi(y)\,dy = \zeta_i(0,0). \qquad \text{(A 3, 35)}$$

Thus for a harbour entrance narrow compared with a wavelength the most important part of the response of the harbour is given by

$$\frac{\zeta(x,y)}{\zeta_i(0,0)} = -\frac{2}{D(k)}\frac{\cos[k(x+d)]}{kb\sin kd}. \qquad \text{(A 4)}$$

$D(k)$ is calculated by a variational approximation and near a minimum modulus at $k = k_0$ is given by

$$D(k) = -\tfrac{1}{2}i + Q\left(1-\frac{k}{k_0}\right) + \dots. \qquad \text{(A 5, 55)}$$

The equation for k_0 is

$$\cot k_0 d = \frac{k_0 b}{\pi}\ln\left(\frac{8}{\gamma k_0 a}\operatorname{cosec}\frac{\pi a}{2b}\right), \qquad \text{(A 6, 56)}$$

where $\ln\gamma$ is Euler's constant. Also

$$Q = \frac{\cot k_0 d}{k_0 b} + \frac{d}{b}\operatorname{cosec}^2 k_0 d - \frac{1}{\pi}. \qquad \text{(A 7, 57)}$$

Clearly for small a/b the second term of Q is dominant (Q is $O(\ln k_0 a)$). The peak response of the harbour occurs very close to k_0, though slightly offset by the variation in $\sin kd$ in the denominator of (A 4), and is given approximately by

$$\frac{\zeta(x,y)}{\zeta_i(0,0)} = -4i(k_0 b k_0 d)^{-\frac{1}{2}} Q^{\frac{1}{2}} \cos[k_0(x+d)]. \tag{A 8}$$

Thus, except for the fundamental mode for which k_0 is very small and the harbour acts as a Helmholtz resonator, the peak response is $O(Q^{\frac{1}{2}})$. The appropriate forcing function for the analogous oscillator is $O(Q^{-\frac{1}{2}})$ and the power transfer factor is $O(1)$. There is a harbour paradox in that as the harbour entrance is decreased, the resonant response to a monochromatic incident wave increases, and the response to a continuous spectrum does not decrease, but this is a weaker result (by a factor $Q^{\frac{1}{2}}$ in the amplitude) than that asserted by Miles & Munk.†

These results are for a rectangular harbour set into a straight coastline, but it seems reasonable to expect the same qualitative results for a harbour of a different shape in the open sea. Thus one might regard an oscillation in such a harbour as a forced oscillator with a forcing function $\nu Q^{-\frac{1}{2}}$ where ν is a factor which is presumably small if the harbour entrance is on the side of the island away from the incoming waves.

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† *Note added in proof.* Miles (1970) has recently developed an elegant general theory of the excitation of oscillations in a harbour of arbitrary shape.